

# Sticky Prices vs. Limited Participation: What Do We Learn From the Data?

Peter Ireland\*

Boston College and NBER

Niki Papadopolou<sup>†</sup>

University of Glasgow

This draft: February 2004

PRELIMINARY AND INCOMPLETE

## Abstract

The method of maximum likelihood is used to estimate a Dynamic Stochastic General Equilibrium business cycle model that combines elements of existing sticky-price and limited-participation specifications. Sticky prices are incorporated, following Rotemberg (1982), by assuming that monopolistically competitive firms face a quadratic cost of nominal price adjustment. Limited participation is incorporated, following Cooley and Quadrini (1999), by assuming that households face a quadratic cost of portfolio adjustment. The results support the hypothesis that the degree of the portfolio adjustment is very small in the data, but significant. In addition, the data suggest that the response of the interest rate to deviations of output from the steady state in the interest rate rule should be very close to zero. This is argued by Christiano and Gust (1999) as well. Furthermore, as in Ireland (1999, 2000), the model can not reject the hypothesis of parameter stability for the policy parameters. On the other hand, the model rejects the hypothesis for the rest of the parameters.

---

\*Please address correspondence to: Peter N. Ireland, Boston College, Department of Economics, Carney Hall, 140 Commonwealth Avenue, Chestnut Hill, MA 02467-3806. Tel: (617) 552-3687. Fax: (617) 552-2308. E-mail: [irelandp@bc.edu](mailto:irelandp@bc.edu).

<sup>†</sup>Please address correspondence to: Niki X. Papadopolou, Department of Economics, University of Glasgow, Adam Smith Building, Room S103, Glasgow G12 8RT, Tel: +44 (0)141 330 5059, Fax: +44 (0)141 330 4940, E-mail: [n.papadopolou@socsci.gla.ac.uk](mailto:n.papadopolou@socsci.gla.ac.uk), URL: <http://www.gla.ac.uk/Acad/PolEcon/papadopolou/>.

# 1. Introduction

In the paper “Sticky Prices, Limited Participation or Both?”, the model that incorporates both sticky prices and limited participation is calibrated based on parameter values that have been used in the literature, and to values that empirically in a similar specification is shown to be giving impulse responses and second moments that can replicate those in the data. A problem though that still remains is the value of the those parameter that other studies have not determined previously e.g. portfolio adjustment costs, and the interest rate rule in a set up that incorporates limited participation.

Secondly, real business cycle models have been criticized that are not as structural to changes in monetary policy regimes they are supposed to be, that the structural parameters cannot remain invariant. This criticism follows the very famous Lucas critique.

Therefore, this paper focuses on the specification and stability of the closed economy dynamic, stochastic, general, equilibrium model that combines sticky prices and limited participation. The model is estimated with maximum likelihood estimation, in order to provide an insight on the degree of the cost of price stickiness and the portfolio adjustment as well as the nature of the interest rate rule, without having any prior assumptions on their level. In addition, the estimation exercise will help us attack the issue of stability as well, in an attempt to show that the structural parameters indeed have remain stable despite the widely believed change in the monetary policy regime that occurred in 1980s.

This methodology, that is proposed in Ireland (1999), combines the dynamic, stochastic, general, equilibrium theory with the flexibility of vector autoregressive time-series models. The purpose is to obtain a hybrid that shares the desirable features of both approaches in macroeconomics: firstly the fact that VARs are designed to be taken directly to the data, are easy to estimate, and can be used for statistical hypothesis tests and forecast analysis, and secondly the fact that DSGE models are based on economic theory. Therefore, the DSGE model is augmented so that its residuals, the movements and co-movements in the data that the theory cannot explain, are described by a VAR allowing us to estimate it and perform hypothesis tests and stability analysis. At the end, the time-series behavior of the endogenous variables is related to the structural parameters that describe private agents’ tastes and technologies.

The results of this paper support the hypothesis that the degree of the portfolio adjustment is very small in the data, but significant. In addition, the data suggest that the response of the interest rate to deviations of output from the steady state in the

interest rate rule should be very close to zero. This is argued by Christiano and Gust (1999) as well.

Turning to the issue of stability, the tests reject the hypothesis of parameter stability of the structural parameters. In addition, the model is not able to account for the instability in the parameters in the policy rule.

The remainder of this paper is organized as follows. Section 2, below, sets up the model. Section 3 describes the data, estimates and tests. Section 4 summarizes and concludes.

## **2. The Model**

### **2.1. Overview**

The model combines elements of existing sticky-price and limited-participation specifications. Sticky prices are incorporated, following Rotemberg (1982), by assuming that monopolistically competitive firms face a quadratic cost of nominal price adjustment. Limited participation is incorporated, following Cooley and Quadrini (1999), by assuming that households face a quadratic cost of portfolio adjustment.

In the model, time periods are indexed by  $t = 0, 1, 2, \dots$ . There are five types of agents: a representative household, a representative finished goods-producing firm, a representative bank, a continuum of intermediate goods-producing firms indexed by  $i \in [0, 1]$ , and a monetary authority. Each intermediate goods-producing firm produces a distinct, perishable intermediate good. Hence, the intermediate goods can also be indexed by  $i \in [0, 1]$ , where good  $i$  is produced by firm  $i$ . Nevertheless, the model contains enough symmetry to allow the analysis to focus on a representative intermediate goods-producing firm, which produces the generic good  $i$ . The activities of each agent are described in the subsections below.

### **2.2. The Representative Household**

The representative household enters period  $t$  with  $M_{t-1}$  units of money and  $K_t$  units of capital. Immediately following the realization of the period- $t$  shocks, the household must decide how to divide its funds into an amount  $D_t$  to be deposited in the representative bank and an amount  $M_{t-1} - D_t$  to be used to facilitate goods purchases. When choosing  $D_t$ , the household faces a quadratic portfolio adjustment cost, measured in terms of time

and given by

$$\tau_t = \frac{\phi_d}{2} \left( \frac{D_t}{\mu D_{t-1}} - 1 \right)^2, \quad (2.1)$$

where  $\phi_d \geq 0$  governs the magnitude of the adjustment cost and where, as noted below,  $\mu \geq 1$  denotes the gross steady-state rate of money growth.

During period  $t$ , the household supplies  $h_t(i)$  units of labor at the nominal wage  $W_t$  and  $K_t(i)$  units of capital at the nominal rental rate  $Q_t$  to each intermediate goods-producing firm  $i \in [0, 1]$ . The household's choices must satisfy

$$h_t = \int_0^1 h_t(i) di,$$

where  $h_t$  denotes total hours worked, and

$$K_t = \int_0^1 K_t(i) di$$

for all  $t = 0, 1, 2, \dots$ .

During period  $t$ , the household purchases output from the representative finished goods-producing firm at the nominal price  $P_t$ . It divides its purchases up into an amount  $C_t$  to be consumed and an amount  $I_t$  to be invested. Since it is assumed that the household receives its wages before making its goods purchases, it faces the cash-in-advance constraint

$$\frac{M_{t-1} - D_t + W_t h_t}{P_t} \geq v_t (C_t + I_t) \quad (2.2)$$

for all  $t = 0, 1, 2, \dots$ . In (2.2),  $v_t$  is a random term that measures the amount of money the household must carry to facilitate its purchases of goods; it is assumed to follow the autoregressive process

$$\ln(v_t) = (1 - \rho_v) \ln(v) + \rho_v \ln(v_{t-1}) + \varepsilon_{vt}, \quad (2.3)$$

where  $v > 0$ ,  $1 > \rho_v > 0$ , and the serially uncorrelated innovation  $\varepsilon_{vt}$  is normally distributed with mean zero and standard deviation  $\sigma_v$ .

By investing  $I_t$  units of the finished good during each period  $t$ , the household increases the capital stock over time according to

$$K_{t+1} = (1 - \delta)K_t + e_t I_t - \frac{\phi_k}{2} \left( \frac{K_{t+1}}{g K_t} - 1 \right)^2 K_t, \quad (2.4)$$

where the depreciation rate satisfies  $1 > \delta > 0$ , where the parameter  $\phi_k \geq 0$  governs the magnitude of capital adjustment costs, and where, as noted below,  $g$  measures the gross

steady-state growth rate of the capital stock. The variable  $e_t$  is Greenwood, Hercowitz, and Huffman's (1988) shock to the marginal efficiency of investment; it follows the autoregressive process

$$\ln(e_t) = \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \quad (2.5)$$

where  $1 > \rho_e > 0$  and the serially uncorrelated innovation  $\varepsilon_{et}$  is normally distributed with mean zero and standard deviation  $\sigma_e$ .

At the end of period  $t$ , the household receives its rental payments  $Q_t K_t$  along with principal plus interest  $r_t^d D_t$  from the bank; hence,  $r_t^d$  measures the gross interest rate on deposits. The household also receives nominal profits  $B_t$  from the representative bank and  $F_t(i)$  from each intermediate goods-producing firm  $i \in [0, 1]$ , for a total of  $B_t + F_t$  in nominal profits, where

$$F_t = \int_0^1 F_t(i) di.$$

The household then carries  $M_t$  units of money into period  $t + 1$ ; it faces the budget constraint

$$\frac{M_{t-1} + (r_t^d - 1)D_t + W_t h_t + Q_t K_t + B_t + F_t}{P_t} \geq C_t + I_t + \frac{M_t}{P_t} \quad (2.6)$$

during each period  $t = 0, 1, 2, \dots$

Thus, the household chooses  $C_t$ ,  $h_t$ ,  $\tau_t$ ,  $D_t$ ,  $M_t$ ,  $I_t$ , and  $K_{t+1}$  for all  $t = 0, 1, 2, \dots$  to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [a_t \ln(C_t) - \gamma(h_t + \tau_t)], \quad (2.7)$$

subject to the constraints imposed by (2.1), (2.2), (2.4), and (2.6) for all  $t = 0, 1, 2, \dots$ . In the utility function,  $1 > \beta > 0$ ,  $\gamma > 0$ , and the preference shock  $a_t$  follows the autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \quad (2.8)$$

where  $1 > \rho_a > 0$  and the serially uncorrelated innovation  $\varepsilon_{at}$  is normally distributed with mean zero and standard deviation  $\sigma_a$ .

Substitute (2.1) into the utility function and (2.4) into the budget and cash-in-advance constraints. Let  $\Lambda_{1t}$  denote the Lagrange multiplier on the budget constraint (2.6) and let  $\Lambda_{2t}$  denote the Lagrange multiplier on the cash-in-advance constraint (2.2). Then the household's first-order conditions include (2.1), (2.2), (2.4), and (2.6) with equality, along with

$$a_t = (\Lambda_{1t} + v_t \Lambda_{2t}) C_t, \quad (2.9)$$

$$\gamma = (\Lambda_{1t} + \Lambda_{2t})(W_t/P_t), \quad (2.10)$$

$$\frac{\Lambda_{1t}}{P_t} = \beta E_t \left( \frac{\Lambda_{1t+1} + \Lambda_{2t+1}}{P_{t+1}} \right), \quad (2.11)$$

$$\begin{aligned} & \gamma \phi_d \left( \frac{D_t}{\mu D_{t-1}} - 1 \right) \frac{D_t}{\mu D_{t-1}} \\ &= \frac{[\Lambda_{1t}(r_t^d - 1) - \Lambda_{2t}]D_t}{P_t} + \beta \gamma \phi_d E_t \left[ \left( \frac{D_{t+1}}{\mu D_t} - 1 \right) \frac{D_{t+1}}{\mu D_t} \right], \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} & (\Lambda_{1t} + v_t \Lambda_{2t})(1/e_t) \left[ 1 + \frac{\phi_k}{g} \left( \frac{K_{t+1}}{gK_t} - 1 \right) \right] \\ &= \beta E_t [\Lambda_{1t+1}(Q_{t+1}/P_{t+1}) + (1 - \delta)(1/e_{t+1})(\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1})] \\ & \quad + \beta \phi_k E_t \left[ (\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1})(1/e_{t+1}) \left( \frac{K_{t+2}}{gK_{t+1}} - 1 \right) \left( \frac{K_{t+2}}{gK_{t+1}} \right) \right] \\ & \quad - \beta (\phi_k/2) E_t \left[ (\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1})(1/e_{t+1}) \left( \frac{K_{t+2}}{gK_{t+1}} - 1 \right)^2 \right] \end{aligned} \quad (2.13)$$

for all  $t = 0, 1, 2, \dots$

### 2.3. The Representative Finished Goods-Producing Firm

During period  $t$ , the representative finished goods-producing firm uses  $Y_t(i)$  units of each intermediate good  $i \in [0, 1]$  to produce  $Y_t$  units of the finished good according to the constant returns to scale technology described by

$$\left[ \int_0^1 Y_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)} \geq Y_t, \quad (2.14)$$

with  $\theta > 1$ . Intermediate good  $i$  sells at the nominal price  $P_t(i)$ , while the finished good sells at the nominal price  $P_t$ ; given these prices, the finished goods-producing firm chooses  $Y_t$  and  $Y_t(i)$  for all  $i \in [0, 1]$  to maximize its profits,

$$P_t Y_t - \int_0^1 P_t(i) Y_t(i) di, \quad (2.15)$$

for each  $t = 0, 1, 2, \dots$

The first-order conditions for this problem can be written as

$$Y_t(i) = [P_t(i)/P_t]^{-\theta} Y_t \quad (2.16)$$

for all  $i \in [0, 1]$  and  $t = 0, 1, 2, \dots$ . Competition in the market for the finished good requires that the representative firm earn zero profits in equilibrium. This zero-profit condition determines  $P_t$  as

$$P_t = \left[ \int_0^1 P_t(i)^{1-\theta} di \right]^{1/(1-\theta)} \quad (2.17)$$

for all  $t = 0, 1, 2, \dots$ .

## 2.4. The Representative Bank

At the beginning of period  $t$ , the representative bank accepts deposits  $D_t$  from the representative household. At the beginning of period  $t$ , the bank also receives a lump-sum nominal transfer  $X_t$  from the monetary authority. Thus, the bank can lend  $L_t(i)$  to each intermediate goods-producing firm  $i \in [0, 1]$ , subject to the constraint

$$D_t + X_t \geq L_t, \quad (2.18)$$

where

$$L_t = \int_0^1 L_t(i) di.$$

At the end of period  $t$ , the bank collects  $r_t L_t(i)$  in principal and interest from each intermediate goods-producing firm  $i \in [0, 1]$ ; hence,  $r_t$  denotes the gross nominal interest rate on loans. Since the bank owes  $r_t^d D_t$  to its depositors, its profits are given by

$$B_t = r_t L_t + D_t + X_t - L_t - r_t^d D_t. \quad (2.19)$$

Competition among banks for loans and deposits guarantees that

$$r_t = r_t^d \quad (2.20)$$

for all  $t = 0, 1, 2, \dots$ . So long as the net nominal interest rate  $r_t - 1$  is positive, the bank will lend out all of its funds and (2.18) will hold with equality.

## 2.5. The Representative Intermediate Goods-Producing Firm

The representative intermediate goods-producing firm hires  $h_t(i)$  units of labor and  $K_t(i)$  units of capital from the representative household during period  $t$  in order to produce  $Y_t(i)$  units of intermediate good  $i$  according to the constant returns to scale technology described by

$$K_t(i)^\alpha [g^t z_t h_t(i)]^{1-\alpha} \geq Y_t(i), \quad (2.21)$$

where  $1 > \alpha > 0$  and where  $g \geq 1$  denotes the gross rate of labor-augmenting technological progress. The aggregate technology shock  $z_t$  follows the autoregressive process

$$\ln(z_t) = (1 - \rho_z) \ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \quad (2.22)$$

where  $z > 0$ ,  $1 > \rho_z > 0$ , and the serially uncorrelated innovation  $\varepsilon_{zt}$  is normally distributed with mean zero and standard deviation  $\sigma_z$ .

The firm rents capital on credit, but must pay its wage bill with funds  $L_t(i)$  borrowed from the representative bank. It therefore faces the finance constraint

$$L_t(i) \geq W_t h_t(i) \quad (2.23)$$

for all  $t = 0, 1, 2, \dots$ . Since these funds are borrowed at the gross rate  $r_t$ , the bank must repay principal plus interest  $r_t L_t(i)$  at the end of the period.

Since intermediate goods substitute imperfectly for one another as inputs to producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market; during each period  $t$ , it sets a nominal price  $P_t(i)$  subject to the requirement that it satisfy the representative finished goods-producing firm's demand, taking  $P_t$  and  $Y_t$  as given.

In addition, each intermediate goods-producing firm faces a quadratic cost of adjusting its nominal price, measured in terms of the finished good and given by

$$\frac{\phi_p}{2} \left[ \frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right]^2 Y_t, \quad (2.24)$$

where  $\phi_p \geq 0$  governs the magnitude of the adjustment cost and where, as noted below,  $\pi \geq 1$  denotes the gross steady-state rate of inflation.

These costs of price adjustment make the firm's problem dynamic; it chooses  $h_t(i)$ ,  $K_t(i)$ ,  $Y_t(i)$ ,  $L_t(i)$ , and  $P_t(i)$  for all  $t = 0, 1, 2, \dots$  to maximize its total market value, equal to

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_{1t} [F_t(i)/P_t], \quad (2.25)$$

where  $\beta^t \Lambda_{1t}/P_t$  represents the marginal utility to the representative household provided by an additional dollar of profits during period  $t$  and where

$$\begin{aligned} F_t(i) = & P_t(i) Y_t(i) + [L_t(i) - W_t h_t(i)] - Q_t K_t(i) \\ & - r_t L_t(i) - \frac{\phi_p}{2} \left[ \frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right]^2 P_t Y_t, \end{aligned}$$



subject to the constraints imposed by its production possibilities, by the finance constraint (2.23), and by the demand curve

$$Y_t(i) = [P_t(i)/P_t]^{-\theta} Y_t$$

for all  $t = 0, 1, 2, \dots$

When the net nominal interest rate  $r_t - 1$  is positive, the finance constraint (2.23) will hold with equality. In this case, the firm's problem simplifies to one of choosing  $h_t(i)$ ,  $K_t(i)$ , and  $P_t(i)$  to maximize its total market value, where

$$\frac{F_t(i)}{P_t} = \left[ \frac{P_t(i)}{P_t} \right]^{1-\theta} Y_t - \frac{Q_t K_t(i) + r_t W_t h_t(i)}{P_t} - \frac{\phi_p}{2} \left[ \frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right]^2 Y_t, \quad (2.26)$$

subject to the single constraint

$$K_t(i)^\alpha [g^t z_t h_t(i)]^{1-\alpha} \geq [P_t(i)/P_t]^{-\theta} Y_t \quad (2.27)$$

for all  $t = 0, 1, 2, \dots$ . The first-order conditions for this problem are (2.27) with equality,

$$\Lambda_{1t} r_t (W_t/P_t) h_t(i) = (1 - \alpha) \Xi_t K_t(i)^\alpha [g^t z_t h_t(i)]^{1-\alpha}, \quad (2.28)$$

$$\Lambda_{1t} (Q_t/P_t) K_t(i) = \alpha \Xi_t K_t(i)^{\alpha-1} [g^t z_t h_t(i)]^{1-\alpha}, \quad (2.29)$$

and

$$\begin{aligned} 0 = & (1 - \theta) \Lambda_{1t} \left[ \frac{P_t(i)}{P_t} \right]^{-\theta} \left( \frac{Y_t}{P_t} \right) + \theta \Xi_t \left[ \frac{P_t(i)}{P_t} \right]^{-\theta-1} \left( \frac{Y_t}{P_t} \right) \\ & - \phi_p \Lambda_{1t} \left[ \frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right] \left[ \frac{Y_t}{\pi P_{t-1}(i)} \right] \\ & + \beta \phi_p E_t \left\{ \Lambda_{1t+1} \left[ \frac{P_{t+1}(i)}{\pi P_t(i)} - 1 \right] \left[ \frac{P_{t+1}(i) Y_{t+1}}{\pi P_t(i)^2} \right] \right\} \end{aligned} \quad (2.30)$$

for all  $t = 0, 1, 2, \dots$ , where  $\Xi_t$  is the Lagrange multiplier on (2.27).

## 2.6. The Monetary Authority

The monetary authority conducts monetary policy by adjusting the short-term nominal interest rate  $r_t$  in response to deviations of detrended output  $y_t = Y_t/g^t$ , inflation  $\pi_t = P_t/P_{t-1}$ , and money growth  $\mu_t = M_t/M_{t-1}$  from their steady-state values  $y$ ,  $\pi$ , and  $\mu$  according to the policy rule

$$\ln(r_t/r) = \rho_y \ln(y_t/y) + \rho_\pi \ln(\pi_t/\pi) + \rho_\mu \ln(\mu_t/\mu) + \varepsilon_{rt}, \quad (2.31)$$

where  $r$  is the steady-state value of  $r$ . In (2.31), the parameters  $\rho_y$ ,  $\rho_\pi$ , and  $\rho_\mu$  should all be positive. The serially uncorrelated innovation  $\varepsilon_{rt}$  is normally distributed with mean zero and standard deviation  $\sigma_r$ .

## 2.7. Symmetric Equilibrium

In a symmetric equilibrium, all intermediate goods-producing firms make identical decisions, so that  $h_t(i) = h_t$ ,  $K_t(i) = K_t$ ,  $F_t(i) = F_t$ ,  $Y_t(i) = Y_t$ ,  $P_t(i) = P_t$ , and  $L_t(i) = L_t$  for all  $i \in [0, 1]$  and  $t = 0, 1, 2, \dots$ . In addition, the market-clearing condition  $M_t = M_{t-1} + X_t$  must hold for all  $t = 0, 1, 2, \dots$ . These equilibrium conditions, together with the first-order conditions for the representative agents' problems, the laws of motion for the aggregate shocks, and the policy rule, form a system of difference equations describing the model's equilibrium. In the absence of shocks, the economy converges to a steady state. The system is log-linearized around its steady state, and the methods of Blanchard and Kahn (1980) can be applied to obtain a solution of the form

$$f_t = U s_t \quad (2.32)$$

and

$$s_t = \Pi s_{t-1} + W \varepsilon_t \quad (2.33)$$

for all  $t = 0, 1, 2, \dots$ .

In (2.32) and (2.33),  $f_t$  is the vector of the model's flow variables which includes output  $y_t = Y_t/g^t$ , inflation  $\pi_t$ , money growth  $\mu_t$ , consumption  $c_t = C_t/g^t$ , investments  $i_t = I_t/g^t$ , the real factor prices  $w_t = (W_t/P_t)/g^t$ , and  $q_t = Q_t/P_t$ , the interest rate  $r_t$ , the nominal transfers  $x_t$ , banks profits  $b_t = B_t/M_t$ , bank loans  $l_t = L_t/M_t$ , hours worked  $h_t$ , real profits  $f_t = (F_t/P_t)/g^t$ , the bank deposits  $d_t = D_t/M_t$ , the multipliers  $\lambda_{1t} = g^t \Lambda_{1t}$ ,  $\lambda_{2t} = g^t \Lambda_{2t}$ , and  $\xi_t = g^t \Xi_t$ .  $s_t$  is the vector of the model's endogenous state variables and the five shocks in the model. The model's endogenous state variables are the lagged values of the bank deposits  $d_{t-1} = D_{t-1}/M_{t-1}$ , the lagged values of real balances  $m_{t-1} = (M_{t-1}/P_{t-1})/g^{t-1}$ , and the current values of the capital stock  $k_t$ . The five shocks in the model are the money demand shock  $v_t$ , the shock to the marginal efficiency of investment  $e_t$ , the preference shock  $a_t$ , the technology shock  $z_t$  and the policy shock  $\varepsilon_{rt}$ . The vector  $\varepsilon_t$  includes the four innovations  $\varepsilon_{vt}$ ,  $\varepsilon_{at}$ ,  $\varepsilon_{zt}$ , and  $\varepsilon_{rt}$  and is assumed to be normally distributed with zero mean and covariance matrix

$$V = E \varepsilon_t \varepsilon_t' = \begin{bmatrix} \sigma_v^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_a^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}$$

The parameters that describe private agents' tastes, technologies and the policy rule determine the elements of the matrices  $\Pi$ ,  $W$ , and  $U$ . The model's solution as described by (2.32) and (2.33) takes the form of a state-space econometric model, driven by the five innovations in  $\varepsilon_t$ . Therefore, maximum likelihood estimations of the parameters in  $\Pi$ ,  $W$ , and  $U$  can be obtained as described in Hamilton (1994) using the Kalman filter together with the data on five variables: consumption  $C_t$ , investment  $I_t$ , money  $M_t$ , prices  $P_t$ , and interest rates  $r_t$ .

### 3. Data, Estimates and Tests

#### 3.1. Data

In the data consumption is measured by personal consumption expenditure, investments are measured by private gross domestic investment, real balances are measured by dividing the M2 money stock by the GDP deflator, inflation is measured by changes in the GDP deflator and the interest rate is measured by the yield on three-month Treasury bills. All series, except for the interest rate, are seasonally adjusted; the series for consumption, investments and real balances are expressed in per-capita terms by dividing by the civilian, noninstitutional population, age 16 and over.

The data are quarterly and run from 1959:1 through 2001:1. The data are divided into two subsamples, the first covering the period 1959:1 through 1979:2, and the second covering the period from 1979:3 through 2001:1. The breakpoint of the sample corresponds to the widely believed change in monetary policy that occurred in 1979:2, when Paul Volker was appointed Chairman of the Board of Governors of the Federal Reserve System.

Distinct upward trends appear in the series for consumption, investments and real balances, because of growth. Ireland (1997) accounts for these trends in the data by including a deterministic trend in the production function that captures the effect of labor-augmenting technological progress. Thus the model implies that  $C_t$ ,  $I_t$ , and  $m_t$  grow at the same rate  $g$  along a balanced growth path.

The data don't contain enough information to estimate all of the model's parameters. Therefore some must be fixed prior to estimation. Thus, the weight  $\gamma$  on leisure is set equal to 1.5, implying that the household spends about one third of its time working. The depreciation rate  $\delta$  is set equal to 0.025. Lastly,  $\theta$  is set equal to 6, implying an average markup of price over marginal cost equal to 20 percent.

### 3.2. Estimates

Table 1 displays maximum likelihood estimates of the model's remaining 20 parameters, together with their standard errors, that are computed by taking the square roots of the diagonal elements of the inverted matrix of second derivatives of the maximized log likelihood function.

The results support the hypothesis that the degree of the portfolio adjustment is statistically significant in the data. In addition, the data suggest that the response of the interest rate to deviations of output from the steady state in the interest rate rule should be very close to zero. This is also argued by Christiano and Gust (1999). Models that incorporate limited participation, should incorporate interest rate rules with the interest rate reacting very little to deviations of output from the steady state in order to have non-explosive results.

Table 2 displays the maximum likelihood estimates for the two subsamples, pre- and post-1980s. It is observed that again the portfolio adjustment cost is significant in both periods, and that the degree of the interest rate response to output deviations from the steady state is very small in both subsamples, especially pre-1980s.

Tables 3 and 4 display the results of the forecast error variance decompositions, in an attempt to find what fraction of the observed consumption and investment variation comes from the five shocks that are incorporated in the model. The estimated model is used to decompose the  $k$ -step-ahead forecast error variances in consumption and investments into five orthogonal components: one attributable to each shock, the money demand, the investment, the preference, the technology and the policy shocks. We observe that for  $k = \infty$  investment shocks account for nearly 99 percent of the unconditional variance in detrended output and investment. For one- to twenty-step-ahead forecast error variances though it is indicated that both preference and technology shocks are those that account for the variation in consumption. Concerning the variation in investments, the shocks that account for its variation for one- to thirty-step-ahead forecast error variance are investment and technology. These results indicate that in addition to technology shocks that are important for the variation of the components of output, consumption is specifically influenced by preference shocks and investment from investment shocks.

### 3.3. Tests

An advantage of the real business cycle models is that they are structural, meaning that they are able to link the behavior of real variables in the economy with private agent's tastes, technologies. These structural parameters, in order to be consistent with the Lucas critique, have to remain invariant to changes in the monetary policy regime.

As discussed above, table 2 displays the maximum likelihood estimates for the two subsamples, pre- and post-1980s. Therefore, the hybrid model can be used to test for parameter stability across the two sub-samples. Andrews and Fair (1988) describe procedures that can be used to test for the stability of the model's estimated parameters across the two subsamples. Let the vector  $\Theta_q^1$  contain  $q$  parameters estimated with pre-1979 data, let  $\Theta_q^2$  contain the same  $q$  parameters estimated with post-1979, and let  $H_q^1$  and  $H_q^2$  denote the covariances matrices of  $\Theta_q^1$  and  $\Theta_q^2$ . Then the Wald statistic can be written more simply as

$$W = (\Theta_q^1 - \Theta_q^2)'(H_q^1 + H_q^2)^{-1}(\Theta_q^1 - \Theta_q^2). \quad (2.34)$$

According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with  $q$  degrees of freedom under the null hypothesis of stability, where  $q$  is the number of parameters being tested for stability.

The Wald statistics in table 5 indicate that the model is not able to accept the null hypothesis that the structural parameters remain stable across the two subsamples. Evidently, there has been a major change in the data between pre- and post- 1980s, and the hybrid model cannot capture its source. On the other hand, the tests indicate that the monetary policy regime has remained stable, since the model can not reject the hypothesis of parameter stability for the policy parameters. This is something puzzling that needs further investigation.

## 4. Conclusions

This paper, focuses on the specification and stability of the estimated model that incorporates sticky prices and limited participation in the financial markets. The model is estimated with maximum likelihood following the methodology in Ireland (1999).

The results indicate that limited participation is statistically significant, although small in the data. In addition, they suggest that the degree of the interest rate response to output deviations from the steady state in the monetary policy rule should be very

small in order to have stability and non-explosive results, something that is argued by Christiano and Gust (1999) as well.

Concerning stability between pre- and post-1980s, where there is believed that a major change in the monetary policy has been occurred, the tests are not able to capture the stability of the structural parameters. Therefore the estimated model is not consistent with the Lucas critique. In addition, the model is not able to account for the instability in the parameters in the policy rule.

## References

- [1] Andrews, Donald W.K., Fair, Ray C. "Inference in Nonlinear Econometric Models with Structural Change." *Review of Economic Studies* 55 (1988): 615-639.
- [2] Blanchard, Olivier J. and Kahn, Charles M. "The Solution of Linear Difference Models Under Rational Expectations." *Econometrica* 48 (1980): 1305-1311.
- [3] Boivin, Jean and Giannoni, Marc. "Has Monetary Policy Become More Effective?" Working Paper 9459. Cambridge: National Bureau of Economic Research, January 2003.
- [4] Christiano, Lawrence J., and Martin Eichenbaum. "Liquidity Effects, and the Monetary Transmission Mechanism." *AEA Papers and Proceedings* 82 No.2 (May 1992): 346-53.
- [5] Christiano, Lawrence J., and Martin Eichenbaum. "Liquidity Effects, Monetary Policy, and the Business Cycle." *Journal of Money, Credit and Banking* 27 (1995): 1113-1136.
- [6] Christiano, Lawrence J., Martin Eichenbaum, Charles L. Evans. "Sticky Price and Limited Participation Models of Money: A Comparison." *European Economic Review* 41 (June 1997): 1201-49.
- [7] Christiano, Lawrence J., Martin Eichenbaum, Charles L. Evans. "Monetary Policy Shocks: What have we Learned And to What End?" *Handbook of Macroeconomics*, ch.2, vol.1.(1999).
- [8] Christiano, Lawrence J., Martin Eichenbaum, Charles L. Evans. "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy." Working Paper 8403. Cambridge: National Bureau of Economic Research, July 2001.
- [9] Christiano, Lawrence J., and Christopher J. Gust. "Taylor Rules in a Limited Participation Model." Working Paper 7017. Cambridge: National Bureau of Economic Research, March 1999.
- [10] Clarida, Richard, Jordi Gali, and Mark Gertler. "Monetary Policy Rules in Practice: Some International Evidence." *European Economic Review* 42 (June 1998): 1033-67.

- [11] Clarida, Richard, Jordi Gali, and Mark Gertler. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory." *The Quarterly Journal of Economics* 115 (February 2000): 147-180.
- [12] Cooley, Thomas F., and Vincenzo Quadrini. "A Neoclassical Model of the Phillips Curve Relation." *Journal of Monetary Economics* 44 (1999): 165-193.
- [13] Dow, James P. Jr. "The Demand and Liquidity Effects of Monetary Shocks". *Journal of Monetary Economics* 36 (1995): 91-115.
- [14] Ellison, Martin and Andrew Scott. "Sticky Prices and Volatile Output." *Journal of Monetary Economics* 46 (December 2000): 621-632.
- [15] Fuhrer, Jeff and George Moore. "Inflation Persistence." *Quarterly Journal of Economics* 110 (February 1995): 127-159.
- [16] Greenwood, Jeremy, Hercowitz, Zvi and Huffman, Gregory W, 1988. "Investment, Capacity Utilization, and the Real Business Cycle." *American Economic Review* 78 (3) (1988):402-17.
- [17] Hamilton, James D. *Time Series Analysis*. Princeton: Princeton University Press, 1994.
- [18] Hendry, Scott and Guang-Jia Zhang. "Liquidity Effects and Market Frictions." *Journal of Macroeconomics* 23 No.1 (Winter 2001): 153-176.
- [19] Ireland, Peter N. "A Small, Structural, Quarterly Model for Monetary Policy Evaluation." *Carnegie-Rochester Conference Series on Public Policy* 47 (1997): 83-108.
- [20] Ireland, Peter N. "A Method for Taking Models to the Data." *Journal of Economic Dynamics and Control*, forthcoming, Working Paper 9903. Federal Reserve Bank of Cleveland (1999)
- [21] Ireland, Peter N. "Interest Rates, Inflation, and Federal Reserve Policy since 1980." *Journal of Money, Credit & Banking* 32 (August 2000): 417-34.
- [22] Ireland, Peter N. "Sticky-Price Models of the Business Cycle: Specification and Stability." *Journal of Monetary Economics* 47 (January 2001): 3-18.
- [23] Keen, Benjamin D. "In Search of the Liquidity Effect in a Modern Monetary Model". Working Paper, University of Virginia (April 2001).



- [24] Kim, Jinill. "Monetary policy in a stochastic equilibrium model with real and nominal rigidities." Finance and Economics Discussion Series, Board of Governors of the Federal Reserve System, 1998-02.
- [25] King, Robert G. and Marc W. Watson. "Money, Prices, Interest Rates and the Business Cycle." *Review of Economics and Statistics* 78 (February 1996): 35-53.
- [26] Lucas, Robert E. "Econometric Policy Evaluation: a Critique." Carnegie-Rochester Conference Series on Public Policy 1 (1976):19-46.
- [27] McCallum, Bennet T. "Analysis of the Monetary Transmission Mechanism: Methodological Issues." Working Paper 7395. Cambridge: National Bureau of Economic Research, October 1999b.
- [28] Rotemberg, Julio J. "Sticky Prices in the United States." *Journal of Political Economy* 90 (December 1982): 1187-1211.
- [29] Rotemberg, Julio J. and Michael Woodford. "Oligopolistic-Based Econometric Framework for the Evaluation of Monetary Policy." *Journal of Political Economy* 100 (December 1992):1153-1207.
- [30] Rotemberg, Julio J. and Michael Woodford. "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy." In Ben S. Bernanke and Julio J. Rotemberg, eds. *NBER Macroeconomics Annual 1997*. Cambridge: MIT Press, 1997.
- [31] Taylor, John B. "Discretion versus Policy Rules in Practice." *Carnegie-Rochester Conference Series on Public Policy* 39 (1993): 195-214.
- [32] Walsh, Carl E. *Monetary Theory and Policy*. Cambridge: The MIT Press, 1998.

Table 1. Full Sample Maximum Likelihood Estimates and Standard Errors

Parameter	Full Sample Estimate	Standard Error
$\beta$	0.9997	0.0005
$\alpha$	0.1685	0.0064
$\phi_d$	0.0275	0.0035
$\phi_k$	4.8556	0.0812
$\phi_p$	25.5239	0.0000
$\mu$	1.0002	0.0128
$\rho_y$	0.0003	0.0015
$\rho_\pi$	0.7832	0.0081
$\rho_\mu$	0.4101	0.0082
$v$	1.0115	0.0091
$z$	1098.9	0.4940
$\rho_v$	0.9999	0.0002
$\rho_e$	1.0000	0.0000
$\rho_a$	0.9673	0.0118
$\rho_z$	0.9063	0.0191
$\sigma_v$	0.0110	0.0006
$\sigma_e$	0.0267	0.0017
$\sigma_a$	0.0099	0.0009
$\sigma_z$	0.0130	0.0008
$\sigma_r$	0.0066	0.0004

Table 2. Subsample Maximum Likelihood Estimates and Standard Errors

Parameter	Pre-1979 Estimate	Standard Error	Post-1979 Estimate	Standard Error
$\beta$	0.9977	0.0005	0.9909	0.0008
$\alpha$	0.2201	0.0065	0.2166	0.0173
$\phi_d$	0.0040	0.0017	0.0290	0.0086
$\phi_k$	7.9017	1.3240	4.9625	0.0127
$\phi_p$	323.8098	3.8239	21.6284	0.0060
$\mu$	1.0613	0.0165	1.0049	0.0113
$\rho_y$	0.0000012	0.0000	0.00046	0.0009
$\rho_\pi$	0.6348	0.0514	0.8321	0.0165
$\rho_\mu$	0.3709	0.0505	0.4002	0.0076
$v$	3.5779	0.5479	0.8210	0.0135
$z$	4043.4	0.9274	50425	0.0343
$\rho_v$	0.9991	0.0011	0.9999	0.0001
$\rho_e$	0.7391	0.0878	1.0000	0.0000
$\rho_a$	0.9656	0.0267	0.9173	0.0335
$\rho_z$	0.9904	0.0041	0.9300	0.0209
$\sigma_v$	0.0105	0.0008	0.0114	0.0010
$\sigma_e$	0.0118	0.0024	0.0210	0.0016
$\sigma_a$	0.0064	0.0005	0.0075	0.0013
$\sigma_z$	0.0184	0.0035	0.0110	0.0012
$\sigma_r$	0.0045	0.0004	0.0041	0.0003

Table 3. Forecast Error Variance Decompositions for Consumption

Quarters Ahead	Full Sample				
	Money Demand	Investment	Preference	Technology	Policy
1	13.9776	8.8818	31.4929	26.6605	18.9872
4	4.4291	3.8451	37.7840	47.0659	6.8759
8	2.4074	2.1296	40.8737	50.8343	3.7550
12	1.8202	1.8423	43.1226	50.3740	2.8409
20	1.3802	3.6297	45.6078	47.2275	2.1548
40	0.9926	16.3689	43.1939	37.8957	1.5488
$\infty$	0.0009	99.921	0.0423	0.0343	0.0014
Quarters Ahead	Pre-1979				
	Money Demand	Investment	Preference	Technology	Policy
1	35.5868	5.1292	4.3401	37.3261	17.6178
4	13.5804	1.3140	3.4016	74.0560	7.6479
8	6.2522	1.0756	2.5695	86.9132	2.9201
12	4.1187	1.0756	2.0867	91.1145	1.6045
20	2.7590	0.6777	1.5115	94.2649	0.7868
40	2.1141	0.3332	0.8670	96.3514	0.3343
$\infty$	11.9625	0.1058	0.2910	87.5387	0.1020
Quarters Ahead	Post-1979				
	Money Demand	Investment	Preference	Technology	Policy
1	19.2401	9.2971	28.3054	32.8098	10.3475
4	6.0060	3.8894	28.6690	57.8700	3.5656
8	3.4320	2.2472	25.7685	66.5350	2.0174
12	2.6628	2.5662	23.2514	69.9720	1.5476
20	2.0491	7.2419	19.2831	70.2611	1.1648
40	1.3918	30.0134	12.7701	55.0696	0.7551
$\infty$	0.0054	99.9668	0.0050	0.0225	0.0003

Table 4. Forecast Error Variance Decompositions for Investment

Quarters Ahead	Full Sample				
	Money Demand	Investment	Preference	Technology	Policy
1	24.5314	24.8265	4.8024	12.2071	33.6327
4	8.7318	32.2586	1.5503	44.0357	13.4235
8	5.2209	36.7312	1.3356	48.6830	8.0293
12	4.3109	41.0208	1.3557	46.6791	6.6335
20	3.7002	47.2553	1.3878	41.9587	5.6981
40	3.2550	53.3663	1.3312	37.0317	5.0158
$\infty$	0.0415	99.4021	0.0171	0.4754	0.0639
Quarters Ahead	Pre-1979				
	Money Demand	Investment	Preference	Technology	Policy
1	40.5289	35.2055	4.1320	1.5548	18.5788
4	21.8284	22.5599	1.5888	43.0170	11.0058
8	10.8738	10.9819	0.6117	72.9083	4.6243
12	7.2419	6.6440	0.3665	83.0233	2.7243
20	4.8920	3.8189	0.2240	89.5011	1.5640
40	3.7724	2.2577	0.1398	92.9079	0.9221
$\infty$	13.8510	1.2548	0.0784	84.3039	49.9613
Quarters Ahead	Post-1979				
	Money Demand	Investment	Preference	Technology	Policy
1	31.2121	32.9433	6.6388	12.1430	17.0627
4	10.2936	41.2419	2.2353	40.1360	6.0931
8	6.0866	45.3085	1.3234	43.6712	3.6104
12	4.9029	49.0295	1.0659	42.0857	2.9160
20	4.0752	54.6030	0.8864	38.0041	2.4313
40	3.5058	60.7106	0.7635	32.9252	2.0950
$\infty$	0.0397	99.7291	0.0049	0.2129	0.0134

Table 5. Tests of Parameter Stability

20 Estimated Parameters	$W = 3189555350^{***}$
5 Policy Parameters: $\mu, \rho_y, \rho_\pi, \rho_\mu, \sigma_r$	$W = 10.0318$
Portfolio Adjustment Cost: $\phi_d$	$W = 8.11331^{***}$
Price Adjustment Cost: $\phi_p$	$W = 6244.97476^{***}$
2 Adjustment Parameters: $\phi_d, \phi_p$	$W = 6349.9029^{***}$
3 Money Demand Parameters: $v, \rho_v, \sigma_v$	$W = 27.6580^{***}$
2 Investment Parameters: $\rho_e, \sigma_e$	$W = 15.6494^{***}$
3 Preference Parameters: $\beta, \rho_a, \sigma_a$	$W = 52.2749^{***}$
3 Technology Parameters: $z, \rho_z, \sigma_z$	$W = 2581737967^{***}$

Note: \*\*\* denotes significance at the 1% level.

# Technical Appendix of “Sticky Prices vs Limited Participation: What Do We Learn From the Data?”

Peter Ireland  
Boston College and NBER

Niki Papadopoulou  
University of Glasgow

February 2004

## 1. Symmetric Equilibrium

In a symmetric equilibrium,  $h_t(i) = h_t$ ,  $K_t(i) = K_t$ ,  $F_t(i) = F_t$ ,  $Y_t(i) = Y_t$ ,  $P_t(i) = P_t$ , and  $L_t(i) = L_t$  for all  $i \in [0, 1]$  and  $t = 0, 1, 2, \dots$ . In addition, the market-clearing condition

$$M_t = M_{t-1} + X_t$$

or

$$\mu_t = 1 + X_t/M_{t-1} \tag{24}$$

must hold for all  $t = 0, 1, 2, \dots$ . It is useful to note that these equilibrium conditions, together with (13)-(15), (17), and (18), can be used to rewrite the household's budget constraint (6) as the aggregate resource constraint

$$Y_t = C_t + I_t + \frac{\phi_p}{2} \left( \frac{\pi_t}{\pi} - 1 \right)^2 Y_t, \tag{6}$$

which must also hold for all  $t = 0, 1, 2, \dots$ .

Collecting and simplifying (1)-(24) yields

$$\tau_t = \frac{\phi_d}{2} \left( \frac{D_t}{\mu D_{t-1}} - 1 \right)^2, \tag{1}$$

$$\frac{M_t}{P_t} = v_t(C_t + I_t), \quad (2)$$

$$\ln(v_t) = (1 - \rho_v) \ln(v) + \rho_v \ln(v_{t-1}) + \varepsilon_{vt}, \quad (3)$$

$$K_{t+1} = (1 - \delta)K_t + e_t I_t - \frac{\phi_k}{2} \left( \frac{K_{t+1}}{gK_t} - 1 \right)^2 K_t, \quad (4)$$

$$\ln(e_t) = \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \quad (5)$$

$$Y_t = C_t + I_t + \frac{\phi_p}{2} \left( \frac{\pi_t}{\pi} - 1 \right)^2 Y_t, \quad (6)$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \quad (7)$$

$$a_t = (\Lambda_{1t} + v_t \Lambda_{2t}) C_t, \quad (8)$$

$$\gamma = (\Lambda_{1t} + \Lambda_{2t})(W_t/P_t), \quad (9)$$

$$\frac{\Lambda_{1t}}{P_t} = \beta E_t \left( \frac{\Lambda_{1t+1} + \Lambda_{2t+1}}{P_{t+1}} \right), \quad (10)$$

$$\begin{aligned} & \gamma \phi_d \left( \frac{D_t}{\mu D_{t-1}} - 1 \right) \frac{D_t}{\mu D_{t-1}} \\ = & \frac{[\Lambda_{1t}(r_t^d - 1) - \Lambda_{2t}]D_t}{P_t} + \beta \gamma \phi_d E_t \left[ \left( \frac{D_{t+1}}{\mu D_t} - 1 \right) \frac{D_{t+1}}{\mu D_t} \right], \end{aligned} \quad (11)$$

$$\begin{aligned} & (\Lambda_{1t} + v_t \Lambda_{2t})(1/e_t) \left[ 1 + \frac{\phi_k}{g} \left( \frac{K_{t+1}}{gK_t} - 1 \right) \right] \\ = & \beta E_t [\Lambda_{1t+1}(Q_{t+1}/P_{t+1}) + (1 - \delta)(1/e_{t+1})(\Lambda_{1t+1} + v_{t+1} \Lambda_{2t+1})] \\ & + \beta \phi_k E_t \left[ (\Lambda_{1t+1} + v_{t+1} \Lambda_{2t+1})(1/e_{t+1}) \left( \frac{K_{t+2}}{gK_{t+1}} - 1 \right) \left( \frac{K_{t+2}}{gK_{t+1}} \right) \right] \\ & - \beta (\phi_k/2) E_t \left[ (\Lambda_{1t+1} + v_{t+1} \Lambda_{2t+1})(1/e_{t+1}) \left( \frac{K_{t+2}}{gK_{t+1}} - 1 \right)^2 \right] \end{aligned} \quad (12)$$

$$D_t + X_t = L_t, \quad (13)$$

$$B_t = r_t X_t, \quad (14)$$

$$r_t = r_t^d, \quad (15)$$

$$\ln(z_t) = (1 - \rho_z) \ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \quad (16)$$



$$L_t = W_t h_t, \quad (17)$$

$$\frac{F_t}{P_t} = Y_t - \frac{Q_t K_t + r_t W_t h_t}{P_t} - \frac{\phi_p}{2} \left( \frac{\pi_t}{\pi} - 1 \right)^2 Y_t, \quad (18)$$

$$Y_t = K_t^\alpha (g^t z_t h_t)^{1-\alpha}, \quad (19)$$

$$\Lambda_{1t} r_t (W_t/P_t) h_t = (1 - \alpha) \Xi_t Y_t, \quad (20)$$

$$\Lambda_{1t} (Q_t/P_t) K_t = \alpha \Xi_t Y_t, \quad (21)$$

$$\begin{aligned} 0 = & (1 - \theta) \Lambda_{1t} + \theta \Xi_t - \phi_p \Lambda_{1t} \left( \frac{\pi_t}{\pi} - 1 \right) \left( \frac{\pi_t}{\pi} \right) \\ & + \beta \phi_p E_t \left[ \Lambda_{1t+1} \left( \frac{\pi_{t+1}}{\pi} - 1 \right) \left( \frac{\pi_{t+1}}{\pi} \right) \left( \frac{Y_{t+1}}{Y_t} \right) \right], \end{aligned} \quad (22)$$

$$\ln(r_t/r) = \rho_y \ln(y_t/y) + \rho_\pi \ln(\pi_t/\pi) + \rho_\mu \ln(\mu_t/\mu) + \varepsilon_{rt}, \quad (23)$$

and

$$\mu_t = 1 + X_t/M_{t-1}. \quad (24)$$

Together with the definitions  $y_t = Y_t/g^t$ ,  $\pi_t = P_t/P_{t-1}$ , and  $\mu_t = M_t/M_{t-1}$ , these 24 equations determine the behavior of the 24 variables  $\tau_t$ ,  $D_t$ ,  $M_t$ ,  $P_t$ ,  $v_t$ ,  $C_t$ ,  $I_t$ ,  $K_t$ ,  $e_t$ ,  $Y_t$ ,  $a_t$ ,  $\Lambda_{1t}$ ,  $\Lambda_{2t}$ ,  $W_t$ ,  $r_t$ ,  $Q_t$ ,  $X_t$ ,  $B_t$ ,  $r_t^d$ ,  $z_t$ ,  $L_t$ ,  $h_t$ ,  $F_t$ , and  $\Xi_t$ .

## 2. Characterizing the Equilibrium

### 2.1. Transformed System

As a first step in solving the model, define the transformed variables  $d_t = D_t/M_t$ ,  $m_t = (M_t/P_t)/g^t$ ,  $\mu_t = M_t/M_{t-1}$ ,  $c_t = C_t/g^t$ ,  $i_t = I_t/g^t$ ,  $k_t = K_t/g^t$ ,  $y_t = Y_t/g^t$ ,  $\lambda_{1t} = g^t \Lambda_{1t}$ ,  $\lambda_{2t} = g^t \Lambda_{2t}$ ,  $w_t = (W_t/P_t)/g^t$ ,  $q_t = Q_t/P_t$ ,  $x_t = X_t/M_{t-1}$ ,  $b_t = B_t/M_t$ ,  $l_t = L_t/M_t$ ,  $f_t = (F_t/P_t)/g^t$ , and  $\xi_t = g^t \Xi_t$ . Use (15) to eliminate  $r_t^d$  from the system, and rewrite (1)-(14) and (16)-(24) as

$$\tau_t = \frac{\phi_d}{2} \left( \frac{\mu_t d_t}{\mu d_{t-1}} - 1 \right)^2, \quad (1)$$

$$m_t = v_t (c_t + i_t), \quad (2)$$

$$\ln(v_t) = (1 - \rho_v) \ln(v) + \rho_v \ln(v_{t-1}) + \varepsilon_{vt}, \quad (3)$$

$$gk_{t+1} = (1 - \delta)k_t + e_t i_t - \frac{\phi_k}{2} \left( \frac{k_{t+1}}{k_t} - 1 \right)^2 k_t, \quad (4)$$

$$\ln(e_t) = \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \quad (5)$$

$$y_t = c_t + i_t + \frac{\phi_p}{2} \left( \frac{\pi_t}{\pi} - 1 \right)^2 y_t, \quad (6)$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \quad (7)$$

$$a_t = (\lambda_{1t} + v_t \lambda_{2t}) c_t, \quad (8)$$

$$\gamma = (\lambda_{1t} + \lambda_{2t}) w_t, \quad (9)$$

$$g\lambda_{1t} = \beta E_t \left( \frac{\lambda_{1t+1} + \lambda_{2t+1}}{\pi_{t+1}} \right), \quad (10)$$

$$\begin{aligned} & \gamma \phi_d \left( \frac{\mu_t d_t}{\mu d_{t-1}} - 1 \right) \frac{\mu_t d_t}{\mu d_{t-1}} \\ = & [\lambda_{1t}(r_t - 1) - \lambda_{2t}] d_t m_t + \beta \gamma \phi_d E_t \left[ \left( \frac{\mu_{t+1} d_{t+1}}{\mu d_t} - 1 \right) \frac{\mu_{t+1} d_{t+1}}{\mu d_t} \right], \end{aligned} \quad (11)$$

$$\begin{aligned} & g(\lambda_{1t} + v_t \lambda_{2t})(1/e_t) \left[ 1 + \frac{\phi_k}{g} \left( \frac{k_{t+1}}{k_t} - 1 \right) \right] \\ = & \beta E_t [\lambda_{1t+1} q_{t+1} + (1 - \delta)(1/e_{t+1})(\lambda_{1t+1} + v_{t+1} \lambda_{2t+1})] \\ & + \beta \phi_k E_t \left[ (\lambda_{1t+1} + v_{t+1} \lambda_{2t+1})(1/e_{t+1}) \left( \frac{k_{t+2}}{k_{t+1}} - 1 \right) \left( \frac{k_{t+2}}{k_{t+1}} \right) \right] \\ & - \beta (\phi_k/2) E_t \left[ (\lambda_{1t+1} + v_{t+1} \lambda_{2t+1})(1/e_{t+1}) \left( \frac{k_{t+2}}{k_{t+1}} - 1 \right)^2 \right] \end{aligned} \quad (12)$$

$$d_t + x_t/\mu_t = l_t, \quad (13)$$

$$b_t \mu_t = r_t x_t, \quad (14)$$

$$\ln(z_t) = (1 - \rho_z) \ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \quad (16)$$

$$m_t l_t = w_t h_t, \quad (17)$$

$$f_t = y_t - q_t k_t - r_t w_t h_t - \frac{\phi_p}{2} \left( \frac{\pi_t}{\pi} - 1 \right)^2 y_t, \quad (18)$$

$$y_t = k_t^\alpha (z_t h_t)^{1-\alpha}, \quad (19)$$

$$\lambda_{1t}r_t w_t h_t = (1 - \alpha)\xi_t y_t, \quad (20)$$

$$\lambda_{1t}q_t k_t = \alpha\xi_t y_t, \quad (21)$$

$$\begin{aligned} 0 = & (1 - \theta)\lambda_{1t} + \theta\xi_t - \phi_p \lambda_{1t} \left( \frac{\pi_t}{\pi} - 1 \right) \left( \frac{\pi_t}{\pi} \right) \\ & + \beta \phi_p E_t \left[ \lambda_{1t+1} \left( \frac{\pi_{t+1}}{\pi} - 1 \right) \left( \frac{\pi_{t+1}}{\pi} \right) \left( \frac{y_{t+1}}{y_t} \right) \right], \end{aligned} \quad (22)$$

$$\ln(r_t/r) = \rho_y \ln(y_t/y) + \rho_\pi \ln(\pi_t/\pi) + \rho_\mu \ln(\mu_t/\mu) + \varepsilon_{rt}, \quad (23)$$

and

$$\mu_t = 1 + x_t. \quad (24)$$

Note also that the definitions of  $\pi_t$ ,  $m_t$ , and  $\mu_t$  imply

$$gm_t \pi_t = \mu_t m_{t-1}. \quad (25)$$

These 24 equations determine the behavior of the 24 stationary variables  $y_t$ ,  $\pi_t$ ,  $\tau_t$ ,  $d_t$ ,  $m_t$ ,  $\mu_t$ ,  $v_t$ ,  $c_t$ ,  $i_t$ ,  $k_t$ ,  $e_t$ ,  $a_t$ ,  $\lambda_{1t}$ ,  $\lambda_{2t}$ ,  $w_t$ ,  $r_t$ ,  $q_t$ ,  $x_t$ ,  $b_t$ ,  $z_t$ ,  $l_t$ ,  $h_t$ ,  $f_t$ , and  $\xi_t$ .

## 2.2. Steady State

In the absence of shocks, the economy converges to a steady state, in which each of the stationary variables is constant. Let  $\mu$  be chosen by policy. Equations (3), (5), (7), and (16) determine  $v$ ,  $e = 1$ ,  $a = 1$ , and  $z$ . Equations (1), (24), and (25) determine

$$\tau = 0,$$

$$\pi = \mu/g,$$

and

$$x = \mu - 1.$$

Equations (10)-(12), (14), and (22) determine

$$r = \pi(g/\beta),$$

$$b = rx/\mu,$$

$$\lambda_2 = (r - 1)\lambda_1,$$

$$q = (g/\beta - 1 + \delta)[1 + v(r - 1)],$$

and

$$\xi = \left( \frac{\theta - 1}{\theta} \right) \lambda_1.$$

Equations (8) and (9) determine

$$c = \frac{1}{\lambda_1 + v\lambda_2}$$

and

$$w = \frac{\gamma}{\lambda_1 + \lambda_2}.$$

Equations (2), (4), (6), (18), (20), and (21) determine

$$k = \frac{c}{(\lambda_1 q / \alpha \xi) - g + 1 - \delta},$$

$$i = (g - 1 + \delta)k,$$

$$m = v(c + i),$$

$$y = \frac{\lambda_1 q k}{\alpha \xi},$$

$$h = \frac{(1 - \alpha)\xi y}{\lambda_1 r w},$$

and

$$f = y - qk - rwh.$$

Equations (13) and (17) determine

$$l = wh/m$$

and

$$d = l - x/\mu.$$

Finally, (19) determines

$$\lambda_1 = \left[ \frac{\gamma}{(1 - \alpha)z} \right] \left( \frac{\theta}{\theta - 1} \right)^{1/(1 - \alpha)} \left( \frac{q}{\alpha} \right)^{\alpha/(1 - \alpha)}.$$

### 2.3. Linearized System

Equations (1)-(14) and (16)-(25) can be log-linearized to describe the behavior of the stationary variables as they fluctuate about their steady-state values in response to shocks. Let  $\hat{y}_t = \ln(y_t/y)$ ,  $\hat{\pi}_t = \ln(\pi_t/\pi)$ ,  $\hat{\tau}_t = \ln(\tau_t/\tau)$ ,  $\hat{d}_t = \ln(d_t/d)$ ,  $\hat{m}_t = \ln(m_t/m)$ ,  $\hat{\mu}_t = \ln(\mu_t/\mu)$ ,  $\hat{v}_t = \ln(v_t/v)$ ,  $\hat{c}_t = \ln(c_t/c)$ ,  $\hat{i}_t = \ln(i_t/i)$ ,  $\hat{k}_t = \ln(k_t/k)$ ,  $\hat{e}_t = \ln(e_t/e)$ ,  $\hat{a}_t = \ln(a_t/a)$ ,  $\hat{\lambda}_{1t} = \ln(\lambda_{1t}/\lambda_1)$ ,  $\hat{\lambda}_{2t} = \ln(\lambda_{2t}/\lambda_2)$ ,  $\hat{w}_t = \ln(w_t/w)$ ,  $\hat{r}_t = \ln(r_t/r)$ ,  $\hat{q}_t = \ln(q_t/q)$ ,  $\hat{x}_t = \ln(x_t/x)$ ,  $\hat{b}_t = \ln(b_t/b)$ ,  $\hat{z}_t = \ln(z_t/z)$ ,  $\hat{l}_t = \ln(l_t/l)$ ,  $\hat{h}_t = \ln(h_t/h)$ ,  $\hat{f}_t = \ln(f_t/f)$ , and  $\hat{\xi}_t = \ln(\xi_t/\xi)$ . Then a log-linear approximation of (1) implies that  $\hat{\tau}_t = 0$ , while log-linear approximations to (2)-(14) and (16)-(25) yield

$$m\hat{m}_t = m\hat{v}_t + v\hat{c}_t + v\hat{i}_t, \quad (2)$$

$$\hat{v}_t = \rho_v \hat{v}_{t-1} + \varepsilon_{vt}, \quad (3)$$

$$gk\hat{k}_{t+1} = (1 - \delta)k\hat{k}_t + i\hat{e}_t + i\hat{i}_t, \quad (4)$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \quad (5)$$

$$y\hat{y}_t = c\hat{c}_t + i\hat{i}_t, \quad (6)$$

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \quad (7)$$

$$\hat{a}_t = \lambda_1 c \hat{\lambda}_{1t} + v \lambda_2 \hat{v}_t + v \lambda_2 c \hat{\lambda}_{2t} + \hat{c}_t, \quad (8)$$

$$0 = \lambda_1 w \hat{\lambda}_{1t} + \lambda_2 w \hat{\lambda}_{2t} + \gamma \hat{w}_t, \quad (9)$$

$$r \hat{\lambda}_{1t} = E_t \hat{\lambda}_{1t+1} + (r - 1) E_t \hat{\lambda}_{2t+1} - r E_t \hat{\pi}_{t+1}, \quad (10)$$

$$\begin{aligned} & \gamma \phi_d \hat{\mu}_t + \gamma \phi_d (1 + \beta) \hat{d}_t - \gamma \phi_d \hat{d}_{t-1} \\ = & \lambda_1 (r - 1) dm \hat{\lambda}_{1t} + \lambda_1 r dm \hat{r}_t - \lambda_2 dm \hat{\lambda}_{2t} \\ & + \beta \gamma \phi_d E_t \hat{\mu}_{t+1} + \beta \gamma \phi_d E_t \hat{d}_{t+1}, \end{aligned} \quad (11)$$

$$\begin{aligned} & g \lambda_1 \hat{\lambda}_{1t} + g v \lambda_2 \hat{v}_t + g v \lambda_2 \hat{\lambda}_{2t} - g (\lambda_1 + v \lambda_2) \hat{e}_t - \phi_k (\lambda_1 + v \lambda_2) \hat{k}_t \\ = & \beta \lambda_1 (q + 1 - \delta) E_t \hat{\lambda}_{1t+1} + \beta \lambda_1 q E_t \hat{q}_{t+1} + \beta (1 - \delta) v \lambda_2 E_t \hat{v}_{t+1} \\ & + \beta (1 - \delta) v \lambda_2 E_t \hat{\lambda}_{2t+1} - \beta (1 - \delta) (\lambda_1 + v \lambda_2) E_t \hat{e}_{t+1} \\ & - \phi_k (1 + \beta) (\lambda_1 + v \lambda_2) \hat{k}_{t+1} + \beta \phi_k (\lambda_1 + v \lambda_2) E_t \hat{k}_{t+2} \end{aligned} \quad (12)$$

$$d\hat{d}_t + (x/\mu)\hat{x}_t - (x/\mu)\hat{\mu}_t = \hat{l}_t, \quad (13)$$

$$\hat{b}_t + \hat{\mu}_t = \hat{r}_t + \hat{x}_t, \quad (14)$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \quad (16)$$

$$\hat{m}_t + \hat{l}_t = \hat{w}_t + \hat{h}_t, \quad (17)$$

$$f\hat{f}_t = y\hat{y}_t - qk\hat{q}_t - qk\hat{k}_t - rwh\hat{r}_t - rwh\hat{w}_t - rwh\hat{h}_t, \quad (18)$$

$$\hat{y}_t = \alpha\hat{k}_t + (1 - \alpha)\hat{z}_t + (1 - \alpha)\hat{h}_t, \quad (19)$$

$$\hat{\lambda}_{1t} + \hat{r}_t + \hat{w}_t + \hat{h}_t = \hat{\xi}_t + \hat{y}_t, \quad (20)$$

$$\hat{\lambda}_{1t} + \hat{q}_t + \hat{k}_t = \hat{\xi}_t + \hat{y}_t, \quad (21)$$

$$\phi_p \hat{\pi}_t = (\theta - 1)\hat{\xi}_t - (\theta - 1)\hat{\lambda}_{1t} + \beta\phi_p E_t \hat{\pi}_{t+1} \quad (22)$$

$$\hat{r}_t = \rho_y \hat{y}_t + \rho_\pi \hat{\pi}_t + \rho_\mu \hat{\mu}_t + \varepsilon_{rt}, \quad (23)$$

$$\mu \hat{\mu}_t = x \hat{x}_t, \quad (24)$$

and

$$\hat{m}_t + \hat{\pi}_t = \hat{\mu}_t + \hat{m}_{t-1}. \quad (25)$$

These 23 equations determine the behavior of the 23 variables  $\hat{y}_t, \hat{\pi}_t, \hat{d}_t, \hat{m}_t, \hat{\mu}_t, \hat{v}_t, \hat{c}_t, \hat{i}_t, \hat{k}_t, \hat{e}_t, \hat{a}_t, \hat{\lambda}_{1t}, \hat{\lambda}_{2t}, \hat{w}_t, \hat{r}_t, \hat{q}_t, \hat{x}_t, \hat{b}_t, \hat{z}_t, \hat{l}_t, \hat{h}_t, \hat{f}_t$ , and  $\hat{\xi}_t$ . In preparing to solve the model, it is convenient to use (25) to rewrite (2) and (17) as

$$m\hat{\mu}_t + m\hat{m}_{t-1} = m\hat{\pi}_t + m\hat{v}_t + vc\hat{c}_t + vi\hat{i}_t \quad (2)$$

and

$$\hat{\mu}_t + \hat{m}_{t-1} + \hat{l}_t = \hat{\pi}_t + \hat{w}_t + \hat{h}_t. \quad (17)$$

It is also convenient to use (3)-(5) to rewrite (12) as

$$\begin{aligned} & g\lambda_1 \hat{\lambda}_{1t} + v\lambda_2 [g - \beta(1 - \delta)\rho_v] \hat{v}_t + gv\lambda_2 \hat{\lambda}_{2t} - \phi_k(\lambda_1 + v\lambda_2) \hat{k}_t \\ & - \{\beta(\lambda_1 + v\lambda_2)[\phi_k(i/k)(1/g) - (1 - \delta)]\rho_e + g(\lambda_1 + v\lambda_2)\} \hat{e}_t \\ = & \beta\lambda_1(q + 1 - \delta)E_t \hat{\lambda}_{1t+1} + \beta\lambda_1 q E_t \hat{q}_{t+1} \\ & + \beta(1 - \delta)v\lambda_2 E_t \hat{\lambda}_{2t+1} + \beta\phi_k(\lambda_1 + v\lambda_2)(i/k)(1/g)E_t \hat{i}_{t+1} \\ & + \phi_k(\lambda_1 + v\lambda_2)\{\beta[(1 - \delta)/g] - (1 + \beta)\} \hat{k}_{t+1} \end{aligned} \quad (12)$$

## 2.4. The Linear System in Matrix Form

Let

$$f_t^0 = \begin{bmatrix} \hat{y}_t & \hat{\pi}_t & \hat{\mu}_t & \hat{c}_t & \hat{i}_t & \hat{w}_t & \hat{r}_t & \hat{q}_t & \hat{x}_t & \hat{b}_t & \hat{l}_t & \hat{h}_t & \hat{f}_t \end{bmatrix}',$$

$$s_t^0 = \begin{bmatrix} \hat{d}_{t-1} & \hat{m}_{t-1} & \hat{k}_t & \hat{d}_t & \hat{\lambda}_{1t} & \hat{\lambda}_{2t} & \hat{\xi}_t \end{bmatrix}',$$

and

$$z_t^0 = \begin{bmatrix} \hat{v}_t & \hat{e}_t & \hat{a}_t & \hat{z}_t & \varepsilon_{rt} \end{bmatrix}'.$$

Then (2), (6), (8), (9), (13), (14), (17)-(21), (23), and (24) can be written as

$$Af_t^0 = Bs_t^0 + Cz_t^0, \quad (26)$$

where  $A$  is  $13 \times 13$ ,  $B$  is  $13 \times 7$ , and  $C$  is  $13 \times 5$ .

Equation (2) implies

$$a_{12} = m$$

$$a_{13} = -m$$

$$a_{14} = vc$$

$$a_{15} = vi$$

$$b_{12} = m$$

$$c_{11} = -m$$

Equation (6) implies

$$a_{21} = y$$

$$a_{24} = -c$$

$$a_{25} = -i$$

Equation (8) implies

$$a_{34} = 1$$

$$b_{35} = -\lambda_1 c$$

$$b_{36} = -v\lambda_2 c$$

$$c_{31} = -v\lambda_2 c$$

$$c_{33} = 1$$

Equation (9) implies

$$a_{46} = \gamma$$

$$b_{45} = -\lambda_1 w$$

$$b_{46} = -\lambda_2 w$$

Equation (13) implies

$$a_{53} = x/\mu$$

$$a_{59} = -x/\mu$$

$$a_{511} = l$$

$$b_{54} = d$$

Equation (14) implies

$$a_{63} = 1$$

$$a_{67} = -1$$

$$a_{69} = -1$$

$$a_{610} = 1$$

Equation (17) implies

$$a_{72} = 1$$

$$a_{73} = -1$$

$$a_{76} = 1$$

$$a_{711} = -1$$

$$a_{712} = 1$$



$$b_{72} = 1$$

Equation (18) implies

$$a_{81} = y$$

$$a_{86} = -rwh$$

$$a_{87} = -rwh$$

$$a_{88} = -qk$$

$$a_{812} = -rwh$$

$$a_{813} = -f$$

$$b_{83} = qk$$

Equation (19) implies

$$a_{91} = 1$$

$$a_{912} = \alpha - 1$$

$$b_{93} = \alpha$$

$$c_{94} = 1 - \alpha$$

Equation (20) implies

$$a_{101} = 1$$

$$a_{106} = -1$$

$$a_{107} = -1$$

$$a_{1012} = -1$$

$$b_{105} = 1$$

$$b_{107} = -1$$

Equation (21) implies

$$a_{111} = 1$$

$$a_{118} = -1$$

$$b_{113} = 1$$

$$b_{115} = 1$$

$$b_{117} = -1$$

Equation (23) implies

$$a_{121} = \rho_y$$

$$a_{122} = \rho_\pi$$

$$a_{123} = \rho_\mu$$

$$a_{127} = -1$$

$$c_{125} = -1$$

Equation (24) implies

$$a_{133} = \mu$$

$$a_{139} = -x$$

Equations (4), (10)-(12), (22), and (25) can be written as

$$DE_t s_{t+1}^0 + FE_t f_{t+1}^0 = Gs_t^0 + Hf_t^0 + Jz_t^0, \quad (27)$$

where  $D$  and  $G$  are  $7 \times 7$ ,  $F$  and  $H$  are  $7 \times 13$ , and  $J$  is  $7 \times 5$ .

Equation (4) implies

$$d_{13} = gk$$

$$g_{13} = (1 - \delta)k$$

$$h_{15} = i$$

$$j_{12} = i$$

Equation (10) implies

$$d_{25} = 1$$

$$d_{26} = r - 1$$

$$f_{22} = -r$$

$$g_{25} = r$$

Equation (11) implies

$$d_{34} = \beta\gamma\phi_d$$

$$f_{33} = \beta\gamma\phi_d$$

$$g_{31} = -\gamma\phi_d$$

$$g_{34} = \gamma\phi_d(1 + \beta)$$

$$g_{35} = -\lambda_1(r - 1)dm$$

$$g_{36} = \lambda_2 dm$$

$$h_{33} = \gamma\phi_d$$

$$h_{37} = -\lambda_1 r dm$$

Equation (12) implies

$$d_{43} = \phi_k(\lambda_1 + v\lambda_2)\{\beta[(1 - \delta)/g] - (1 + \beta)\}$$

$$d_{45} = \beta\lambda_1(q + 1 - \delta)$$

$$d_{46} = \beta(1 - \delta)v\lambda_2$$

$$f_{45} = \beta\phi_k(\lambda_1 + v\lambda_2)(i/k)(1/g)$$

$$f_{48} = \beta\lambda_1 q$$

$$g_{43} = -\phi_k(\lambda_1 + v\lambda_2)$$

$$g_{45} = g\lambda_1$$

$$g_{46} = gv\lambda_2$$

$$j_{41} = v\lambda_2[g - \beta(1 - \delta)\rho_v]$$

$$j_{42} = -\{\beta(\lambda_1 + v\lambda_2)[\phi_k(i/k)(1/g) - (1 - \delta)]\rho_e + g(\lambda_1 + v\lambda_2)\}$$

Equation (22) implies

$$f_{52} = \beta\phi_p$$

$$g_{55} = \theta - 1$$

$$g_{57} = 1 - \theta$$

$$h_{52} = \phi_p$$

Equation (25) implies

$$d_{62} = 1$$

$$g_{62} = 1$$

$$h_{62} = -1$$

$$h_{63} = 1$$

The presence of  $d_t$  in  $s_{t+1}^0$  and  $s_t^0$  implies

$$d_{71} = 1$$

$$g_{74} = 1$$

Finally, (3), (5), (7), and (16) can be written as

$$z_{t+1}^0 = Pz_t^0 + \varepsilon_{t+1}, \tag{28}$$

where

$$P = \begin{bmatrix} \rho_v & 0 & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 & 0 \\ 0 & 0 & \rho_a & 0 & 0 \\ 0 & 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{vt} & \varepsilon_{et} & \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}'.$$

### 3. Solving the Model

Rewrite (26) as

$$f_t^0 = A^{-1}Bs_t^0 + A^{-1}Cz_t^0,$$

and substitute it into (27) to obtain

$$E_t s_{t+1}^0 = K s_t^0 + L z_t^0, \quad (29)$$

where

$$K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$$

and

$$L = (D + FA^{-1}B)^{-1}(J + HA^{-1}C - FA^{-1}CP).$$

If the  $7 \times 7$  matrix  $K$  has three eigenvalues inside the unit circle and four eigenvalues outside the unit circle, then the system has a unique solution. If  $K$  has more than four eigenvalues outside the unit circle, then the system has no solution. If  $K$  has less than four eigenvalues outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980).

Assuming from now on that there are exactly four eigenvalues outside the unit circle, write  $K$  as

$$K = M^{-1}NM,$$

where

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

The diagonal elements of  $N$  are the eigenvalues of  $K$ , with those in the  $3 \times 3$  matrix  $N_1$  inside the unit circle and those in the  $4 \times 4$  matrix  $N_2$  outside the unit circle. The columns of  $M^{-1}$  are the eigenvectors of  $K$ ;  $M_{11}$  is  $3 \times 3$ ,  $M_{12}$  is  $3 \times 4$ ,  $M_{21}$  is  $4 \times 3$ , and  $M_{22}$  is  $4 \times 4$ . In addition, let

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where  $L_1$  is  $3 \times 5$  and  $L_2$  is  $4 \times 5$ .

Now (29) can be rewritten as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} E_t s_{t+1}^0 = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s_t^0 + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} z_t^0$$

or

$$E_t s_{1t+1}^1 = N_1 s_{1t}^1 + Q_1 z_t^0 \quad (30)$$

and

$$E_t s_{2t+1}^1 = N_2 s_{2t}^1 + Q_2 z_t^0, \quad (31)$$

where

$$s_{1t} = M_{11} \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + M_{12} \begin{bmatrix} \hat{d}_t \\ \hat{\lambda}_{1t} \\ \hat{\lambda}_{2t} \\ \hat{\xi}_t \end{bmatrix}, \quad (32)$$

$$s_{2t} = M_{21} \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + M_{22} \begin{bmatrix} \hat{d}_t \\ \hat{\lambda}_{1t} \\ \hat{\lambda}_{2t} \\ \hat{\xi}_t \end{bmatrix}, \quad (33)$$

$$Q_1 = M_{11}L_1 + M_{12}L_2,$$

and

$$Q_2 = M_{21}L_1 + M_{22}L_2.$$

Since the eigenvalues in  $N_2$  lie outside the unit circle, (31) can be solved forward to obtain

$$s_{2t} = -N_2^{-1} R z_t^0,$$

where the  $4 \times 5$  matrix  $R$  is given by

$$\begin{aligned} \text{vec}(R) &= \text{vec} \sum_{j=0}^{\infty} N_2^{-j} Q_2 P^j = \sum_{j=0}^{\infty} \text{vec}(N_2^{-j} Q_2 P^j) \\ &= \sum_{j=0}^{\infty} [P^j \otimes (N_2^{-1})^j] \text{vec}(Q_2) = \sum_{j=0}^{\infty} (P \otimes N_2^{-1})^j \text{vec}(Q_2) \\ &= [I_{(20 \times 20)} - P \otimes N_2^{-1}]^{-1} \text{vec}(Q_2). \end{aligned}$$

Use this result, along with (33), to solve for

$$\begin{bmatrix} \hat{d}_t \\ \hat{\lambda}_{1t} \\ \hat{\lambda}_{2t} \\ \hat{\xi}_t \end{bmatrix} = S_1 \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + S_2 z_t^0, \quad (34)$$

where

$$S_1 = -M_{22}^{-1}M_{21}$$

and

$$S_2 = -M_{22}^{-1}N_2^{-1}R.$$

Equation (32) now provides a solution for  $s_{1t}^1$ :

$$s_{1t} = (M_{11} + M_{12}S_1) \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + M_{12}S_2 z_t^0.$$

Substitute this result into (30) to obtain

$$\begin{bmatrix} \hat{d}_t \\ \hat{m}_t \\ \hat{k}_{t+1} \end{bmatrix} = S_3 \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + S_4 z_t^0, \quad (35)$$

where

$$S_3 = (M_{11} + M_{12}S_1)^{-1}N_1(M_{11} + M_{12}S_1)$$

and

$$S_4 = (M_{11} + M_{12}S_1)^{-1}(Q_1 + N_1M_{12}S_2 - M_{12}S_2P).$$

Finally, return to

$$\begin{aligned} f_t^0 &= A^{-1}Bs_t^0 + A^{-1}Cu_t \\ &= A^{-1}B \begin{bmatrix} I_{(3 \times 3)} \\ S_1 \end{bmatrix} \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + A^{-1}B \begin{bmatrix} 0_{(3 \times 5)} \\ S_2 \end{bmatrix} z_t^0 + A^{-1}Cz_t^0, \end{aligned}$$

which can be written more simply as

$$f_t^0 = S_5 \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + S_6 z_t^0, \quad (36)$$

where

$$S_5 = A^{-1}B \begin{bmatrix} I_{(3 \times 3)} \\ S_1 \end{bmatrix}$$

and

$$S_6 = A^{-1}B \begin{bmatrix} 0_{(3 \times 5)} \\ S_2 \end{bmatrix} + A^{-1}C.$$

Equations (28) and (34)-(36) provide the model's solution:

$$s_{t+1} = \Pi s_t + W \varepsilon_{t+1} \quad (37)$$

and

$$f_t = U s_t, \quad (38)$$

where

$$\begin{aligned} s_t &= \begin{bmatrix} \hat{d}_{t-1} & \hat{m}_{t-1} & \hat{k}_t & \hat{v}_t & \hat{e}_t & \hat{a}_t & \hat{z}_t & \hat{\varepsilon}_{rt} \end{bmatrix}', \\ f_t &= \begin{bmatrix} \hat{y}_t & \hat{\pi}_t & \hat{\mu}_t & \hat{c}_t & \hat{i}_t & \hat{w}_t & \hat{r}_t & \hat{q}_t & \hat{x}_t & \hat{b}_t & \hat{l}_t & \hat{h}_t & \hat{f}_t & \hat{d}_t & \hat{\lambda}_{1t} & \hat{\lambda}_{2t} & \hat{\xi}_t \end{bmatrix}', \\ \varepsilon_t &= \begin{bmatrix} \varepsilon_{vt} & \varepsilon_{et} & \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}', \end{aligned}$$

$$\Pi = \begin{bmatrix} S_3 & S_4 \\ 0_{(5 \times 3)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(3 \times 5)} \\ I_{(5 \times 5)} \end{bmatrix},$$

and

$$U = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix}.$$

## 4. Estimating the Model

Suppose that data are available on consumption  $C_t$ , investment  $I_t$ , money  $M_t$ , prices  $P_t$ , and interest rates  $r_t$ . These data can be used to construct a series  $\{d_t\}_{t=1}^T$ , where

$$d_t = \begin{bmatrix} \hat{c}_t \\ \hat{i}_t \\ \hat{m}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \ln(C_t) - t \ln(g) - \ln(c) \\ \ln(I_t) - t \ln(g) - \ln(i) \\ \ln(M_t) - \ln(P_t) - t \ln(g) - \ln(m) \\ \ln(P_t) - \ln(P_{t-1}) - \ln(\pi) \\ \ln(r_t) - \ln(r) \end{bmatrix}.$$



Equations (37) and (38) then given rise to an empirical model of the form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \quad (39)$$

and

$$d_t = Cs_t, \quad (40)$$

where  $A = \Pi$ ,  $B = W$ ,  $C$  is formed from the rows of  $\Pi$  and  $U$  as

$$C = \begin{bmatrix} U_4 \\ U_5 \\ \Pi_2 \\ U_2 \\ U_7 \end{bmatrix},$$

and the vector of serially uncorrelated innovations

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{vt+1} & \varepsilon_{et+1} & \varepsilon_{at+1} & \varepsilon_{zt+1} & \varepsilon_{rt+1} \end{bmatrix}'$$

is assumed to be normally distributed with zero mean and diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}' = \begin{bmatrix} \sigma_v^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_a^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}.$$

The model defined by (39) and (40) is in state-space form; hence, the likelihood function for the sample  $\{d_t\}_{t=1}^T$  can be constructed as outlined by Hamilton (1994, Ch.13). For  $t = 1, 2, \dots, T$  and  $j = 0, 1$ , let

$$\hat{s}_{t|t-j} = E(s_t | d_{t-j}, d_{t-j-1}, \dots, d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t | d_{t-j}, d_{t-j-1}, \dots, d_1).$$

Then, in particular, (39) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(8 \times 1)} \quad (41)$$

and

$$vec(\Sigma_{1|0}) = vec(Es_1s_1') = [I_{(64 \times 64)} - A \otimes A]^{-1}vec(BVB'). \quad (42)$$

Now suppose that  $\hat{s}_{t|t-1}$  and  $\Sigma_{t|t-1}$  are in hand and consider the problem of calculating  $\hat{s}_{t+1|t}$  and  $\Sigma_{t+1|t}$ . Note first from (40) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_tu_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned} \hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'] [E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1} u_t \\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t. \end{aligned}$$

Hence, from (39),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Using this last result, along with (39) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t | d_{t-1}, d_{t-2}, \dots, d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_tu_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t | d_{t-1}, d_{t-2}, \dots, d_1),$$

$$Eu_t u_t' = C\Sigma_t C' = \Omega_t,$$

the sequences for  $K_t$  and  $\Sigma_t$  can be generated recursively using

$$K_t = A\Sigma_t C' (C\Sigma_t C')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C' (C\Sigma_t C')^{-1} C\Sigma_t A',$$

and the initial conditions  $\hat{s}_1$  and  $\Sigma_1$  are provided by (41) and (42).

The innovations  $\{u_t\}_{t=1}^T$  can then be used to form the log likelihood function for  $\{d_t\}_{t=1}^T$  as

$$\ln L = -\frac{5T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \ln |\Omega_t| - \frac{1}{2} \sum_{t=1}^T u_t' \Omega_t^{-1} u_t.$$

The model's 24 parameters are

$$\begin{aligned}
\phi_d &\geq 0 \\
\mu &\geq 1 \\
v &> 0 \\
1 > \rho_v &\geq 0 \\
\sigma_v &> 0 \\
1 > \delta &> 0 \\
\phi_k &\geq 0 \\
g &\geq 1 \\
1 > \rho_e &\geq 0 \\
\sigma_e &> 0 \\
1 > \beta &> 0 \\
\gamma &> 0 \\
1 > \rho_a &\geq 0 \\
\sigma_a &> 0 \\
\theta &> 1 \\
1 > \alpha &> 0 \\
z &> 0 \\
1 > \rho_z &\geq 0 \\
\sigma_z &> 0 \\
\phi_p &\geq 0 \\
\rho_y &> 0 \\
\rho_\pi &> 0 \\
\rho_\mu &> 0 \\
\sigma_r &> 0
\end{aligned}$$

## 5. Evaluating the Model

### 5.1. Testing for Parameter Stability

The procedures described by Andrews and Fair (1988) can be used to test for the stability of the model's estimated parameters. Let  $\Theta^1$  and  $\Theta^2$  denote the

estimated parameters from two disjoint subsamples, and let  $H^1$  and  $H^2$  denote the associated covariance matrices, so that asymptotically,

$$\Theta^1 \sim N(\Theta^{10}, H^1)$$

and

$$\Theta^2 \sim N(\Theta^{20}, H^2).$$

One way of testing for the stability of all of the estimated parameters is with the likelihood ratio statistic

$$LR = 2[\ln L(\Theta^1) + \ln L(\Theta^2) - \ln L(\Theta)],$$

where  $\ln L(\Theta^1)$ ,  $\ln L(\Theta^2)$ , and  $\ln L(\Theta)$  are the maximized log likelihood functions for the first subsample, the second subsample, and the third entire sample. According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with  $q$  degrees of freedom under the null hypothesis of stability, where  $q$  is the number of estimated parameters.

Alternatively, the stability of some or all of the parameters can be tested with the Wald statistic

$$W = g(\Theta^1, \Theta^2)'(G\hat{H}G')^{-1}g(\Theta^1, \Theta^2),$$

when the stability restrictions are written as

$$g(\Theta^1, \Theta^2) = 0$$

and where

$$G = \frac{\vartheta g(\Theta^1, \Theta^2)}{\vartheta(\Theta^1, \Theta^2)'}$$

and

$$\hat{H} = \begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix}.$$

If  $\Theta_q^1$  and  $\Theta_q^2$  denote the subsets of  $\Theta^1$  and  $\Theta^2$  of interest, and if  $H_q^1$  and  $H_q^2$  denote the covariances matrices of  $\Theta_q^1$  and  $\Theta_q^2$ , then this Wald statistic can be written more simply as

$$W = (\Theta_q^1 - \Theta_q^2)'(H_q^1 + H_q^2)^{-1}(\Theta_q^1 - \Theta_q^2).$$

According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with  $q$  degrees of freedom under the null hypothesis of stability, where  $q$  is the number of parameters being tested for stability.

## 5.2. Variance Decompositions

Begin by considering (39), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t$$

or

$$(1 - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B \varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B \varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B \varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B \varepsilon_{t+k-j},$$

and hence

$$\begin{aligned} \Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\ &= BV B' + ABV B' A' + A^2 BV B' A^{2'} + \dots + A^{k-1} BV B' A^{k-1'}. \end{aligned}$$

In addition, (39) implies that

$$\Sigma^s = \lim_{k \rightarrow \infty} \Sigma_k^s$$

is given by

$$vec(\Sigma^s) = [I_{(64 \times 64)} - A \otimes A]^{-1} vec(BVB').$$

Next, consider (38) and (39), which imply that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U \Sigma_k^s U',$$

$$\Sigma^f = \lim_{k \rightarrow \infty} \Sigma_k^f = U \Sigma^s U',$$

$$\Sigma_k^d = E(d_{t+k} - E_t d_{t+k})(d_{t+k} - E_t d_{t+k})' = C \Sigma_k^s C',$$

and

$$\Sigma^d = \lim_{k \rightarrow \infty} \Sigma_k^d = C \Sigma^s C'.$$

Let  $\Theta$  denote the vector of estimated parameters, and let  $H$  denote the covariance matrix of these estimated parameters, so that asymptotically,

$$\Theta \sim N(\Theta^0, H).$$

Note that the elements of  $\Sigma_k^s$ ,  $\Sigma^s$ ,  $\Sigma_k^f$ ,  $\Sigma^f$ ,  $\Sigma_k^d$ , and  $\Sigma^d$  can all be expressed as nonlinear functions of  $\Theta$  :

$$\Sigma = g(\Theta),$$

so that asymptotic standard errors for these elements can be found by calculating

$$\nabla g H \nabla g'.$$

In practice, the gradient  $\nabla g$  can be evaluated numerically, as suggested by Runkle (1987).